Image Segmentation by the Level Set Methods Using Third Order WENO

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Abstract: All level set based image segmentation methods are based on an assumption that the level set function is or close to a signed distance function (SDF). Small time step and costly re-initialization procedure must be applied to guarantee this assumption, and in order to calculate the gradient, simple numerical schemes, based on finite differences, are applied.

In this paper, in order to achieve higher order accuracy in the temporal discretization, we have used Total Variation Diminishing (TVD) Runge Kutta (RK) methods. The spatial derivatives are determined by using the Weighted Essentially Non-Oscillatory methods (WENO-3) that accurately capture the formation of sharp gradients in the moving fronts. Experiments results show that we have obtained good results both on synthetic and real images.

Key words: Active contour, PDE, ENO and WENO schemes, level set.

INTRODUCTION

Segmentation processing is a crucial step in image analysis. It consists mainly in detecting and visualizing the common boundaries of distinct objects in the image.

Nowadays, the popular approaches are the active contour models. These models are widely used in many applications, including edge detection; shape modelling and motion tracking. The first model is the classical snake formulated by Kass & al [MKA 87].

It is a method of surrounding the boundary of an object in an image by closed curve. In this model a closed curve deforms under the influence of internal forces, image forces and external constraint forces.

But this model has its drawbacks. For example, it is sensitive to initial curve position and initial curve shape and doesn't have the ability of changing with topology.

The use of partial differential equations (PDE) in image processing and computer vision, in particular, the use of level set method and dynamic implicit surfaces, has increased dramatically in recent years. Traditionally these closely fields, image processing and computer vision, have developed independently. However, level set and related PDE-based methods have served to provide new set of tools that have led to increased interaction.

The level set method overcomes the drawbacks of snake model. It is a numerical and theoretical tool for propagating interfaces. It was first introduced by Osher & Sethian[SOS 88], and has become a more and more popular theoretical and numerical framework within image processing, fluid mechanics, graphics, computer vision, etc. The level set approach is able to handle complex topological changes automatically.

1. Front propagation and link with hyperbolic conservation laws

Consider a curve moving in a plane. Let γ(0) be the initial curve. γ(t) is obtained by moving γ(0) along its normal with a speed F, where F is a function of the curvature κ.
Figure 1 Front propagation with speed $F$.

Let $X(s,t) = (x(s,t), y(s,t))$ be the position vector which parameterizes $\gamma(t)$ by $s$, $0 \leq s \leq S$, $X(0,t)=X(S,t)$

The curve is parameterized so that the interior is on the left in the direction of increasing $s$. The equation of motion can be written as [SOS 88],[JAS 85],[JAS 96]:

$$
\begin{align*}
    x_t &= F(\kappa) \frac{y}{(x^2+y^2)^{3/2}}, \\
    y_t &= -F(\kappa) \frac{x}{(x^2+y^2)^{3/2}}
\end{align*}
$$

(1)

To be solved for $t \in [0, \infty)$ with $X(s,0)=\gamma(0)$.

The curvature $\kappa$ is defined by

$$
\kappa = \frac{y_x x_y - x_x y_y}{(x^2+y^2)^{3/2}}
$$

(2)

We can also define an inverse mapping function $f$ defined by $t = f(x,y)$. The function $f$ satisfies the partial differential equation

$$
\frac{\partial u}{\partial t} + F(\kappa) \frac{\partial u}{\partial x} = 0
$$

as long as the curve $\gamma$ stays smooth and non-intersecting.

If we consider a small section of the curve $t = f(x,y)$ which we can write as $y = \psi(x,t)$, we arrive (after inserting it in the previous relation and taking $F(\kappa) = 1 - \varepsilon \kappa$) at the following equation

$$
\psi_t = \left(1 + \psi_x^2\right)^{3/2} \left(1 + \frac{\varepsilon \psi_t}{1 + \psi_x^2}\right)^{\varepsilon/2}
$$

(3)

Which is a Hamilton-Jacobi equation with second order viscosity [SOS 88].

Let $u = \psi_x$ and taking the $x$ derivative of the previous equation, we obtain

$$
u_t + [G(u)]_x = \varepsilon \left(1 + \psi_x^2\right)^{3/2} \frac{u_x}{G(u)^2}
$$

(4)

for $G(u) = \left(1 + u^2\right)^{3/2}$, $G(u)$ concave. We can see that the characteristics propagate into shocks ($\varepsilon = 0$) that is to corners in equation (4) corresponding to $F = 1$. The weak solution in this case is not unique, and an additional entropy condition is needed to select the correct viscosity solution limit [GUY 85].

In the previous sections, there is an assumption that the front stays smooth, but in many cases, the smoothness is soon lost. A famous example is the cosine curve propagating with unit speed $F(\kappa) = 1$ (No viscosity) and $F(\kappa) = 1 - \varepsilon \kappa$ (some viscosity)

Figure 2 Cosine Curve Propagating with Speed $F(\kappa)$.

It is well known that for $\varepsilon = 0$, shock discontinuities can develop in the solution, even for smooth initial data. We can see in Fig 2.a that the front develops a sharp corner in finite time.

Conversely for $\varepsilon > 0$, the viscosity term diffuses the steepening fronts and the solution remains smooth. For example, in the case of Burgers equation without viscosity ($\varepsilon = 0$), an entropy condition is used to select the proper way of continuing the solution past the point when the shock develops, resulting in a globally defined weak solution which is the limiting solution of equation as $\varepsilon \to 0$.

2. Level set methods

Consider a curve moving in a plane. Let $\gamma(0)$ be the initial curve. $\gamma(t)$ is obtained by moving $\gamma(0)$ along its normal with a speed $F$ which may depend on local properties (such as curvature and normal direction), global properties of the front (such as integrals along the front or associated differential equations), and independent properties (such as underlying fluid velocity). The central idea in the level set approach is to represent the front $\gamma(t)$ as the level set $\{\Phi = 0\}$ of a higher dimensional function $\Phi$.

2.1. Implicit function and signed distance function

There are various implicit functions which can be used to represent interface. In particular, we used a signed distance function to implicitly represent the interface. It is defined as follows[XUD 94]:

$$
\begin{align*}
    \phi < 0 \quad &\text{if} \quad x \in \Omega^- \\
    \phi = 0 \quad &\text{if} \quad x \in \partial \Omega \\
    \phi > 0 \quad &\text{if} \quad x \in \Omega^+
\end{align*}
$$

(5)
Figure 3 *Convex regions* \((\kappa > 0)\) and *Concave regions* \((\kappa < 0)\).

The interface is represented by the function \(\Phi=0\), which is also called zero level set function.

The level set function is defined as the signed distance function. The value is the distance to the nearest point on the front which is negative inside and positive outside (of course zero at the boundaries). The existence of the front means that the signed distance level set function has positive and negative parts. This property should be kept through the iterations in order not to lose the front. There are several approaches in the literature for the re-initialization of the level set function.

The level set function is usually updated by the following equation:

\[
\Phi_t = \text{sgn}(\Phi) \left( 1 - \left| \nabla \Phi \right| \right)
\]  

Solving this equation frequently often keeps the function with a gradient magnitude equal to one at the steady state \((\left| \nabla \Phi \right| = 1)\).

2.2. Initial Formulation

In the previous subsection, we presented the representation for the interface, which is denoted as the zero isocontour of a function \(\Phi\). We can link the evolution of this function to the propagation of the front itself through a time dependent initial value problem. At any time, the front is given by the zero level set of the time-dependent level set function \(\Phi\).

Then we get the following equation [JAS 96]:

\[
\Phi(x(t),t) = 0
\]

Taking the time derivative for both sides of the equation, by the chain rule, we get

\[
\Phi_t + \Phi(x(t),t) x'(t) = 0
\]  

Since \(F\) is defined as the speed in the outward normal direction, then \(x'(t)n = F\), where \(n = \nabla \Phi / \left| \nabla \Phi \right|\).

This yields an evolution equation for \(\Phi\):

\[
\Phi_t + F \left| \nabla \Phi \right| = 0 \quad \text{given} \quad \Phi(x,t=0)
\]  

This is the level set equation given by Osher and Sethian [SOS 88].

By using this equation, the level set method can handle topological changes naturally. Fig. 4 illustrates the topological change phenomenon.

2.3. Deformable models based on curve shortening

Alvarez & al [LAL 92] described an algorithm for image selective smoothing and edge detection. In this case, the image evolves according to

\[
I_t = g\left(\left\| G * \nabla I \right\| \right) \left( \nabla I \cdot \frac{\nabla I}{\left| \nabla I \right|} \right)
\]

Where \(G\) is a smoothing kernel (for example a Gaussian), and \(g(\omega)\) is a non-increasing function which tends to zero as \(\omega \to \infty\).

We note that \(\left\| \nabla I \right\| \left( \nabla I \cdot \frac{\nabla I}{\left| \nabla I \right|} \right)\) is equal to \(I_{xx}\).

Where \(\xi\) is the direction normal to \(\nabla I\). Thus it diffuses \(I\) in the direction orthogonal to the gradient \(\nabla I\), and does not diffuse in the direction of \(\nabla I\). This means that the image is being smoothed on both side of the edge, with minimal smoothing at the edge itself.

We note again that the evolution

\[
I_t = \left( \nabla I \cdot \nabla I \right) = \kappa \left| \nabla I \right|
\]

is such that the level sets of \(I\) move according to the Euclidean shortening flow [CHI 97].
Finally the term \( g\left(\|G \ast \nabla I\|\right) \) is used for the enhancement of the edges. If \( \|\nabla I\| \) is small then the diffusion is strong. If \( \|\nabla I\| \) is large at a certain point \((x,y)\), this point is considered as an edge point, and the diffusion is weak.

The equation (10) represents, of course, an anisotropic diffusion.

If we assume that the deforming curve \( C \) is given as a level set of a function \( \Phi \), then we can represent the deformation of \( C \) via the deformation of \( \Phi \). The proposed deformation is obtained by modifying the edge detection algorithm (10) and by including an inflationary force in the normal direction governed by a positive real constant \( \nu \).

The evolution equation takes the form

\[
\frac{\partial \Phi}{\partial t} = \nu \left(\nabla \Phi \right) \cdot \left(\nabla \nabla \Phi\right) + g \left(\nabla \Phi\right)
\]

\[
\Phi_{t} = \nu \left(\nabla \Phi \right) \cdot \left(\nabla \nabla \Phi\right) + g \left(\nabla \Phi\right)
\]

(12)

Where, \( \nu \) is the curvature of level set function \( \Phi \), and the stopping term typically has the form

\[
g = \frac{1}{1 + \|\nabla \Phi\|^2}
\]

\( p = 1 \) or \( 2 \).

\( \hat{I} \) is a regularized version (filtered image) of the original image I.

We remind that the curvature can be written as:

\[
\kappa = \frac{\Phi_{xx} \Phi_{yy} - 2 \Phi_{xy} \Phi_{xy} + \Phi_{x} \Phi_{y} + \Phi_{y} \Phi_{x}}{\left\| \nabla \Phi \right\|^{3}}
\]

(13)

2.4. Numerical Discretization

A Hamilton-Jacobi equation has a form:

\[
\Phi_{t} + H(\nabla \Phi) = 0
\]

(14)

Of course, the level set equation is an example of a Hamilton Jacobi equation \(( H(\nabla \Phi) = F(\nabla \Phi) = -g(\nabla \nabla \Phi)\).

It is also known that the right approach to obtain a solution is to study the equations written in conservation form. But in this case, we have to define the numerical schemes that agree with this form. These schemes are called conservative schemes.

Let us consider a general hyperbolic equation of conservation laws:

\[
\nu_{t} + \frac{1}{2} \frac{\partial}{\partial x} (\pi(v)) = 0
\]

(15)

\[
\nu(0,x) = v_{0}(x)
\]

We say that the numerical scheme is in conservative form if it can be written as

\[
u^{n}_{t,i} = u^{n}_{i} - \frac{\Delta t}{\Delta x} \left[ \Pi \left( \Delta_{i-1} \cdots \Delta_{i+n} \right) - \Pi \left( \Delta_{i-1} \cdots \Delta_{i+n} \right) \right]
\]

(16)

For some function \( \Pi \) of \((p+q+1)\) arguments, \( \Pi \) is called the numerical flux function. Of course some consistency relations between \( \Pi \) and \( \pi \) have to be satisfied. It was proved that the numerical scheme is consistent with the original conservation law if \( \Pi \) reduces to \( \pi \) for the case of constant solution \( \nu(t,x) = c \), then necessary

\[
\Pi(c,c) = \pi(c) \forall c \in \mathbb{R}
\]

(17)

This notion is important because, according to the Lax-Wendroff theorem, if the numerical scheme is consistent and in a conservative form, and if the resulting sequence of approximated solutions converges, then necessarily the limiting function is a weak solution of the conservation law.

The geodesic active contour model \( \Phi_{t} = g \left(\kappa + \nu \right) \left\| \nabla \Phi \right\| \) consists of two terms \( g \kappa \left\| \nabla \Phi \right\| \) and \( g \nu \left\| \nabla \Phi \right\| \). We notice that the first term represents a parabolic equation and has diffusive effects, so the use of upwind schemes are inappropriate, and classical central differences are used.

The discrete scheme of the equation (13) is as follows:

\[
\Phi_{t,i,j}^{n+1} = \Phi_{t,i,j}^{n} + \Delta t \left\{ g_{i,j}^{n} \left( \delta_{x}^{+} \Phi_{t,i,j}^{n} \right)^{2} + \left( \delta_{y}^{+} \Phi_{t,i,j}^{n} \right)^{2} \right\}^{\frac{1}{2}}
\]

(18)

\[
+ \nu \left[ \max \left( g_{i,j,0}^{n} \right) \left( \nabla^{+} \Phi_{t,i,j}^{n} \right) + \min \left( g_{i,j,0}^{n} \right) \left( \nabla^{-} \Phi_{t,i,j}^{n} \right) \right]
\]

where

\[
\nabla^{+} \Phi_{t,i,j}^{n} = \left[ \max \left( \delta_{x}^{+} \Phi_{t,i,j}^{n} \right)^{2} + \min \left( \delta_{x}^{+} \Phi_{t,i,j}^{n} \right)^{2} \right]^{\frac{1}{2}}
\]

and

\[
\delta_{x}^{+} u^{n}_{i,j} = \frac{u^{n}_{i+1,j} - u^{n}_{i,j}}{\Delta x}, \quad \delta_{y}^{+} u^{n}_{i,j} = \frac{u^{n}_{i,j+1} - u^{n}_{i,j}}{\Delta y}
\]

and

\[
\delta_{x}^{-}, \delta_{y}^{-}, \delta_{x}, \delta_{y}
\]

are respectively the forward, backward and centered schemes.

\( \nabla^{-} \Phi_{t,i,j}^{n} \) is obtained from \( \nabla^{+} \Phi_{t,i,j}^{n} \) by inverting the signs plus and minus.

For the temporal discretization, we can view
the equation (13) as a nonlinear evolution operator of the type
\[ \frac{\partial}{\partial t} \Phi_{jk} = -L(\Phi, j, k) \] (19)

To achieve higher order accuracy in the temporal discretization, one can use Total Variation Diminishing (TVD) Runge Kutta (RK) methods.

These methods guarantee that the total variation of the solution does not increase, so that no new extrema are generated. Using a TVD method would ensure that spurious oscillations caused by the numerical method don’t occur. The 2nd order TVD RK which is also known as the midpoint rule is given by

\[ \Phi_{i+\frac{1}{2}} = \Phi_{i-\frac{1}{2}} + \frac{\Delta t}{2} [ \Phi_{i+1} - \Phi_{i} ] \]

To find \( \Phi_{i+\frac{1}{2}} \) we start with \( k = i+1 \) and to find \( \Phi_{i-\frac{1}{2}} \) we start with \( k = i \). Then we define

\[ \Phi_{i+\frac{1}{2}} (x_i) = Q_i (x_i) + Q_{i+1} (x_i) + Q_{i+2} (x_i) \]

To find \( \Phi_{i-\frac{1}{2}} \) we start with \( k = i-1 \) and to find \( \Phi_{i-\frac{1}{2}} \) we start with \( k = i \). Then we define

\[ \Phi_{i-\frac{1}{2}} (x_i) = Q_i (x_i) + Q_{i-1} (x_i) + Q_{i-2} (x_i) \]

Improvements are obtained by including the \( Q_i \) and \( Q_{i+k} \) terms leading to second and third order accuracy respectively.

Looking at the difference table and noting that \( D_{i+\frac{1}{2}} \Phi \) was chosen for the first order accuracy, we have two choices for the second order accurate correction. We could include the next point to the left and use \( D_{i-\frac{1}{2}} \Phi \) or we could include the next point to the right and use \( D_{i+\frac{1}{2}} \Phi \). But if \( |D_{i+k} \Phi| \leq |D_{i-k} \Phi| \) we set \( c = D_{i+k} \Phi \) and \( k = k-1 \). Otherwise we set \( c = D_{i-k} \Phi \) and \( k = k \). Then we define

\[ Q_i (x_i) = c (x_i - x_{i-k}) (x_i - x_{i-k+1}) \]

so that \( Q_i (x_i) = c (i-1) \Delta x \) is the second order accurate correction to the approximation of \( \Phi_i \).

Similar to the second order, the third order accurate correction is obtained by comparing \( D_{i+\frac{1}{2}} \Phi \) and \( D_{i+\frac{1}{2}} \Phi \). If \( |D_{i+\frac{3}{2}} \Phi| \leq |D_{i+\frac{1}{2}} \Phi| \) we set

\[ c' = D_{i+\frac{1}{2}} \Phi \]

otherwise we set \( c' = D_{i+\frac{3}{2}} \Phi \). Then we define

\[ Q_i (x) = c' (3i^2 k^2 - 6i k^2 + 2k^2 + 2) \Delta x \]

is the third order accurate correction to the approximation of \( \Phi_i \) [SOS 03].

Weighted ENO (WENO) methods combine the results obtained using all possible stencils rather than choosing only one. A weighted combination of the results from all stencils is used, where the weight are based on the magnitudes of the divided
differences in such a way that smoother approximations receive greater weight. This is more robust than placing all the weight on a single stencil, since it responds more smoothly to changes in the data. Further details can be found in [XUD 94][GUA 96][FRE 03].

3. Results

We have applied our approach on a variety of both synthetic and real images; experiments results show that we have achieved satisfactory results.

Figure 5 deals with the segmentation of a synthetic image which has geometrical shapes without noise.

The initial level set function is \( \phi_0(x, y) \) representing several (100) circles. The curve is reinitialized every 5 steps and \( \nu=1 \). For the spatial discretization we have used a third order WENO scheme. We can see that the curve evolves quickly, and after 200 iterations, it surrounds the two rectangles shapes. The curve in 3D represents the distance function.

Figure 6 deals with a real image of two cells. The edges are obtained at 60 iterations.

Figure 7 deals with a real image of carotid. The edges are obtained at 400 iterations.

Figure 8 deals with a radiographic image of weld. The edges are obtained at 270 iterations.

Figure 5 Segmentation of an image with several circles.

Figure 6 Real image of two cells.
we have used Total Variation Diminishing (TVD) Runge Kutta (RK) methods. The spatial derivatives are determined by using the Weighted Essentially Non-Oscillatory methods (WENO-3). Experiments show that we have obtained satisfactory results both on synthetic and real images.

REFERENCES


4. Conclusion

In this article, we have presented an image segmentation based on level set methods. The level set approach is able to handle complex topological changes automatically. In order to achieve higher order accuracy in the temporal discretization,