Completion of Covariance Matrix
via Four-order Moment with Power Control
for Partially Augmentable Array

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Abstract: Some methods such as convex projection algorithm based on maximum entropy (CP-ME) were presented to implement the completion of covariance matrix for partially augmentable array. However, the conventional methods are complicated and costs large computation. In the letter, a new method is proposed to implement the completion of covariance matrix for partially augmentable array. By utilizing signals’ four-order moment based on power control technique, the new method becomes much easier and faster than CP-ME which only takes use of signals’ t wo-order moment. Experimental result proves new method to be effective.

Key words: four-order moment, partially augmentable array, power control

1 Introduction
Comparing with fully augmentable array (FAA), partially augmentable array (PAA) has an incomplete difference set of intersensor spacings, so it cannot obtain a complete covariance matrix of corresponding uniform linear array (ULA) of the same aperture by applying direct augmentation approach (DAA) as FAA. Some methods such as convex projection algorithm based on maximum entropy (CP-ME) are presented to implement the completion of covariance matrix for PAA [1-2]. However, the methods are complicated and costs much more computation than DAA. Even like that, it is still significant to study the completing methods for PAA, because PAA achieves larger aperture, namely higher resolution than FAA of equivalent sensors in DOA estimation. Since CP-ME only takes use of signals two-order moment, here, signals four-order moment together with power control technique is considered to implement the completion of covariance matrix for PAA. The new method is easier and faster than the methods in [1-2], and it is also proved to be effective by experimental result.

2 Completion of covariance matrix via
four-order moment with power control
Consider \( Q \) narrow-band remote plane-wave signals \( x_q(t) \), \( q = 1, \ldots, Q \) from azimuth bearing \( \theta = [\theta_1, \theta_2, \ldots, \theta_Q] \) impinging on a linear array of \( M \) identical omni-directional sensors located at integer position \( D = [d_1, d_2, \ldots, d_M] \) measured in wavelength units. Signals are independent of zero mean and power \( P_q \). With power control technique applied successfully, each \( P_q \) is assumed to be same \( P = P_q \). The output of the \( m \) th sensor is

\[
x_m(t) = \sum_{q=1}^{Q} s_q(t) \cdot e^{j2\pi d_m \sin \theta_q} + n_m(t) \tag{1}
\]

where \( m = 1, \ldots, M \), \( n_m(t) \) is additive white Gaussian noise of the \( m \) th sensor with zero mean and covariance \( \sigma^2 \), uncorrelated with each other and independent of signals. Equation 1 can be written as

\[
X = AS + N \tag{2}
\]

where

\[
X = [x_1(t), x_2(t), \ldots, x_M(t)]^T, \\
S = [s_1(t), s_2(t), \ldots, s_Q(t)]^T, \\
N = [n_1(t), n_2(t), \ldots, n_M(t)]^T, \\
A = [a_1, a_2, \ldots, a_M]^T
\]

is array manifold with column

\[
a_m = [a_{m,1}, a_{m,2}, \ldots, a_{m,Q}]
\]

called steering vector,
\[ a_{m,q} = \exp(-j2\pi \cdot d_m \cdot \sin \theta_q) \]. \( T \) denotes transpose.

From equation 2, we get the covariance matrix

\[ R_x = E[X \cdot X'^H] = A R_s A'^H + \sigma^2 I \quad (3) \]

where \( H \) denotes conjugate transpose, diagonal matrix \( R_s = \text{diag}(P_1, P_2, \ldots, P_q) \) is signal variance matrix, \( \sigma^2 I_q \) is \( Q \times Q \) identity matrix while \( I \) is \( M \times M \) identity matrix. As for \( k_1, k_2 = 1, \ldots, M \), the element located in row \( k_1 \) and column \( k_2 \) of \( R_x \) is

\[
\begin{align*}
& r_{k_1,k_2} = E[x_{k_1} \cdot x_{k_2}'] \\
& = \sum_{q=1}^{Q} P \cdot e^{j2\pi d_{m,q} \cdot \sin \theta_q} + \delta(k_1 - k_2) \cdot \sigma^2 \quad (4)
\end{align*}
\]

where \( \delta(k) \) is Dirac function, equaling to 1 if \( k = 0 \) and 0 otherwise.

If the array is a FAA, e.g. \( D = [0,2,5,8,9] \), the difference set of intersensor spacings \( \{ d_m = d_{m,k} - d_{m,j} | k, j = 1, \ldots, 5 \} = \{0,1,2,3,4,5,7,8,9\} \) is complete. If the array is a PAA, e.g. \( D = [0,1,4,9,11] \), the corresponding set \( \{ d_m = d_{m,k} - d_{m,j} | k, j = 1, \ldots, 5 \} = \{0,1,2,3,4,5,7,8,9,10,11\} \) is incomplete with \( \overline{d_m} = \pm 6 \). Assume missing lags. Generally, \( |\overline{d_m}| > 1 \).

Define a four-order moment of array outputs as below, \( l_1, l_2, l_3, l_4 = 1, \ldots, M \)

\[
\rho_{l_1,l_2,l_3,l_4} = E[x_{i}(t)x'_{j}(t)x'_{k}(t)x_{l}(t)) \quad (5)
\]

With assumption of signals and noise above, we have

\[
\begin{align*}
& \rho_{l_1,l_2,l_3,l_4} = \sum_{q=1}^{Q} P^2 \cdot e^{j2\pi d_{m,q} \cdot \sin \theta_q} + \\
& \Gamma(l_1, l_2, l_3, l_4) \cdot \sigma^4
\end{align*}
\]

where

\[
\begin{align*}
& \Gamma(l_1, l_2, l_3, l_4) = \delta(l_1 - l_2) \cdot \delta(l_3 - l_4) + \\
& \delta(l_1 - l_2) \cdot \delta(l_3 - l_4) \cdot \delta(l_1 - l_3)
\end{align*}
\]

Analogically, let

\[
\{ d_m = d_{m,k} - d_{m,j} + d_{m,l} | k, j, l = 1, \ldots, M \}
\]

denotes the difference set of four-sensor spacings. Then, for PAA \( D = [0,1,4,9,11] \), there exist the following combination satisfying \( d_m = \overline{d_m} = \pm 6 \)

\[ \pm (9 - 4 + 1 - 0) = \pm 6 \quad , \quad \pm (11 - 4 + 0 - 1) = \pm 6 \quad . \]

\[ \pm (4 - 1 + 4 - 1) = \pm 6 \quad , \quad \pm (11 - 9 + 4 - 0) = \pm 6 \quad . \]

Refer to equation 4 and 6, the missing covariance lag \( \overline{d_m} \) in equation 4 can be obtained easily from equation 6 where \( d_m = \overline{d_m} \). Since generally \( |d_m = \overline{d_m}| > 1 \), we have \( \Gamma(l_1, l_2, l_3, l_4) = 0 \) in equation 6, thus

\[
\begin{align*}
& \rho(d_m = \overline{d_m}) = \sum_{q=1}^{Q} P^2 \cdot e^{j2\pi d_{m,q} \cdot \sin \theta_q} \\
& \quad \text{therefore, the new method for the completion of missing covariance lag} \quad \overline{d_m} \quad \text{of PAA can be expressed as}
\end{align*}
\]

\[
\begin{align*}
& r(\overline{d_m}) = \sum_{q=1}^{Q} P \cdot e^{j2\pi d_{m,q} \cdot \sin \theta_q} = \rho(\overline{d_m})/P \quad (7)
\end{align*}
\]

For convenience, here names the new method as FOM-PC. Undoubtedly, \( d_m \) has much more combinations than \( \overline{d_m} \), hence it nearly always exists some combinations to implement the completion for PAA with equation 7. For example, for one missing lag \( \overline{d_m} > 1 \), either \( \overline{d_m} + 1 \) or \( \overline{d_m} - 1 \) exists. Assume \( \overline{d_m} = 1 = d_{h,k} - d_{k,l} \) or \( \overline{d_m} = -1 = d_{h,k} - d_{k,l} \), and

\[
\begin{align*}
& 1 = d_{h,k} - d_{k,l} \\
& \rho(\overline{d_m} = 1) = d_{h,k} - d_{k,l} + d_{k,l} - d_{h,k} \\
& \rho(\overline{d_m} = -1) = d_{h,k} - d_{k,l} + d_{k,l} - d_{h,k}
\end{align*}
\]

3 Simulation

Assume 6 signals of \( \theta = [-19^\circ, -10^\circ, 1^\circ, 11^\circ, 21^\circ, 29^\circ] \) with equivalent power impinging on the PAA \( D = [0,1,4,9,11] \), sensor’s SNR=10dB, snapshots=300. Experimental results with two methods are shown in Fig.1 and Fig.2 respectively. Fig.1 corresponds to MUSIC method with DAA which causes a partial completion, Fig.2 corresponds to MUSIC method with FOM-PC which leads to a fully completion. Experimental results prove that FOM-PC is effective for PAA. The asterisk * in the figures denotes the true DOA.

![Fig.1 MUSIC method with DAA](image-url)
Conclusion
A new method making use of signals four-order moment combined with power control technique is proposed to implement the completion of covariance matrix for partially augmentable array. Comparing to the traditional methods in [1-2], the new method is easy and fast, and also effective proved by experimental result.

Reference