A New Adaptive Anisotropic Diffusion using the Local Intensity Variance

Y. Zaz, L. Masmoudi, K. Bouzouba and L. Radouane

LESSI, DEPARTEMENT DE PHYSIQUE, FACULTÉ DES SCIENCES B.P 1796-30000 FÉZ, MOROCCO

Abstract: This paper presents a new Adaptive Anisotropic Diffusion technique – based on Perona-Malik (P-M) equation - to smooth a signal while preserving discontinuities. In P-M equation, We propose to adapt the Scale Dimension k parameter of the diffusivity function according to the local intensity variance, which is computed for each image pixel. We will show that this technique removes noise and irrelevant details while preserving sharper boundaries. The results show that the proposed approach outperforms classical P-M technique.

Key words: Adaptive anisotropic diffusion, Perona-Malik equation, Scale Dimension k parameter, local intensity variance.

1 Introduction

Image segmentation and edge detection are two fundamental procedures of computer vision that rely on image smoothing as an important first step. Their goal is to decompose a given image into regions that are essentially homogeneous (with little variation in brightness). These regions should be separated by sharp boundaries (edges). In order to do that, anisotropic diffusion was introduced in image processing as an image enhancement method (Perona & Malik 1990), (Catte, Coll, Lions, & Morel 1992), (Rekeczky, Roska, Ushida 1998) and (Abd-Elmoniem, Youssef, & Kadah 2002). Almost all of those methods inspire from the P-M equation (Perona & Malik 1990), which stimulated a great deal of interest in image processing community (Guo 1999), (Nitzberg & Shiota 1992) and (Black, Sapiro, Marimont, & Heeger 1998). It is commonly believed that the P-M equation provides a potential algorithm for image segmentation, noise removing, edge detection, and image enhancement. The basic idea behind the P-M algorithm is to evolve an original image, under an edge-controlled diffusion operator (see section 2), (Saint-Marc, Chen & Medieni 1991) mentioned the problem of this equation is in the choice of the parameter of the diffusivity function: If k is chosen to be large, all the discontinuities disappear, and the result is the same as if Gaussian smoothing was used. If k is chosen to be small, then all the discontinuities are preserved, and no smoothing is performed. In order to solve this problem, we propose in this paper an adaptive determination of this parameter k. The adaptation is computed according to the local intensity variance.

The paper is organized as follows; the isotropic and anisotropic diffusion will be briefly reviewed in section 2. In section 3 study of conductivity function, the computation of the adaptive k parameter, and the algorithm of the new adaptive anisotropic diffusion are described. Section 4 contains experimental results; the results show that the new method outperforms P-M method. Finally, the section concludes the paper.

2 Isotropic and anisotropic diffusion

The idea behind the use of diffusion equation in image processing arose from the use of the Gaussian filter in multi-scale image analysis. Convolving an image with a Gaussian filter $G_s$:

$$G_s(x,y)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2+y^2}{2\sigma^2}} \quad (1)$$

with standard deviation $s$. (Koenderink 1984) showed that this convolution is equivalent to the solution of the diffusion equation in...
two dimensions:

\[
\begin{align*}
\frac{\partial I(x, y, t)}{\partial t} &= \frac{\partial^2 I(x, y, t)}{\partial x^2} + \frac{\partial^2 I(x, y, t)}{\partial y^2} \\
I(x, y, 0) &= I_0(x, y)
\end{align*}
\]  

(2)

Where \( I_0 \) is the original image.

This isotropic diffusion has some important disadvantages. Firstly, filtering an image with Gaussian smoothes both noise and other unwanted features as well as important features (like edges) which makes identification harder (Weickert 1998). Secondly edges are dislocated when smoothing an image at coarser scales (Weickert 1998) which makes it harder to relate edges detected at coarse scales with edges at finer scales. To overcome those disadvantages P-M (Perona & Malik 1990) and (Perona, Shiota & Malik 1994) propose the anisotropic diffusion: coarser scales of smoothing is used in image areas of similar intensity values, while finer smoothing scales is used at edges and at features where the gradient is strong (i.e. where the intensity values locally differ rapidly). They proposed an implementation using an anisotropic diffusion equation:

\[
\begin{align*}
\frac{\partial I(x, y, t)}{\partial t} &= \text{div}(c(\|\nabla I(x, y, t)\|)\nabla I(x, y, t)) \\
I(x, y, 0) &= I_0(x, y)
\end{align*}
\]  

(3)

Where \( c(.) \) is the conductivity function (or diffusivity function), \( \nabla \) is the gradient operator, and \( \text{div} \) is the divergence operator. If \( c(.) \) is a constant leads to a linear diffusion equation, with an homogeneous diffusivity. In this case, all locations in the image, including the edges are smoothed equally.

One motivation of the work in (Perona & Malik 1990) and (Perona, Shiota & Malik 1994) is achieving both noise removal and edge enhancement through the use of an equation (3). Discretized form of this equation is (Black, Sapiro, Marimont, & Heeger 1998):

\[
I_{s+1}^t = I_s^t + \frac{\tau}{|\mathcal{N}_s|} \sum_{p \in \mathcal{N}_s} c(\|\nabla I_{s,p}\|)I_{s,p}
\]  

(4)

Where \( I_s^t \) is a discretely sampled image, \( s \) denotes the pixel position in a discrete, two-dimensional (2-D) grid, and \( t \) denotes discrete time steps (iterations). The constant \( \tau \in \mathbb{R}^+ \) is a scalar that determines the rate of diffusion, \( |\mathcal{N}_s| \) represents the spatial neighborhood of pixel \( s \), and \( |\mathcal{N}_s| \) is the number of neighbors (usually four). Perona and Malik linearly approximated the image gradient magnitude in a particular direction as:

\[
\|\nabla I_{s,p}\| = I_p - I_s^t \quad p \in \mathcal{N}_s
\]  

(5)

Perona and Malik suggested two different conductivity functions \( c(.) \):

\[
c\|\nabla I\| = e^{-\frac{\|\nabla I\|}{k}}
\]  

(6)

and

\[
c\|\nabla I\| = \frac{1}{1 + \frac{\|\nabla I\|^2}{k}}
\]  

(7)

3 The adaptive anisotropic diffusion filter.

3.1 Study of conductivity function

The function curve of equation [6] is shown in the following figure

![Figure 1. Evolution of conductivity function for different value of k parameter.](image)

As we see, this function is chosen to give a value equal to one (i.e. maximum diffusion) at low values of the gradient magnitude, and values of zero (i.e. no diffusion) for strong gradient. This will preserve edges while blurring regions of small intensity values.

\[
\begin{align*}
&\text{for } \|\nabla I\| \rightarrow \infty \quad c \rightarrow 0 \\
&\text{for } \|\nabla I\| \rightarrow 0 \quad c \rightarrow 1
\end{align*}
\]

The problem of the equation 6 as mentioned in (Saint-Marc, Chen & Medieni 1991) is: If \( k \) is chosen to be large, all discontinuities disappear, and the result is the same as if Gaussian smoothing was used. If \( k \) is chosen to be small, then all the discontinuities are preserved, and no smoothing is performed. It’s can be seen clearly by the inspection of figure 1. If \( k \) is large, the conductivity takes values near 1 where the gradient magnitude \( \|\nabla I\| \) takes medium values. And if \( k \) is small, the conductivity tends to zero where the gradient magnitude is on the medium values.

To solve this problem we propose an adaptive determination of \( k \) parameter according to the local intensity variance.
3.2 Adaptive computation of k parameter.

The k parameter is determined using this proposed formulation:

\[ k_{i,j} = k_{\text{max}} - \left( \frac{V(i,j) \cdot k_{\text{max}} - k_{\text{min}}}{V_{\text{max}}} \right) \]  

(8)

Where \( k_{\text{max}} \) and \( k_{\text{min}} \) are the maximal and minimal of k, respectively (designed by the user).

And \( V(i,j) \) is the local intensity variance and \( V_{\text{max}} \) is maximal value of the variance.

This variance is used in previous work (Zaz, Masmoudi, Bouzouba, & Radouane 2004). It’s computed as follows:

- At each I(i,j) pixel, for a kernel size \((2L+1) \times (2L+1)\) centered at \((i,j)\), and for the four principal directions, calculate the difference \(d_i\) between the received image intensity averages over S1 and S2 regions (Figure 2).

\[ d_i = \frac{1}{(2L+1)^2} \left[ \sum_{(i,j) \in S_1} I(i,j) - \sum_{(i,j) \in S_2} I(i,j) \right] \]  

(9)

Evaluate the four differences \(d_i\) and retain the direction \(p\), which corresponds to the difference \(dp\) of maximal value.

\[ d_p = \max\{d_i\} \quad \text{with} \quad i = 1..4 \]  

(10)

- For the direction \(p\), compute the local intensity error variance \(V_1\) on S1 and \(V_2\) on S2, then

\[ V(i,j) = V_1(i,j) + V_2(i,j) \]  

(11)

with \( V_1(i,j) = \sum_{(i,j) \in S_1} (I(i,j) - E(i,j))^2 \)  

(12)

and \( V_2(i,j) = \sum_{(i,j) \in S_2} (I(i,j) - E(i,j))^2 \)  

(13)

\( E(i,j) \) is the fitted step defined:

by: \( E(i,j) = \begin{cases} 
  m - \beta p h & \text{if } (i,j) \in S_1 \\
  m + \beta p h & \text{if } (i,j) \in S_2 
\end{cases} \)  

(14)

where \( \beta p \) is determined adaptively as:

\[ \beta p = \begin{cases} 
  0 & d_p > 0 \\
  -1 & \text{otherwise} 
\end{cases} \]  

(15)

\( h \) is a positive parameter which defines the jump of mean and \(m\) is the minimum of local error variance and is defined by

\[ m = \frac{\sum_{(i,j) \in S_1} (I(i,j) + \beta p h) + \sum_{(i,j) \in S_2} (I(i,j) - \beta p h)}{(2L+1)^2} \]  

(16)

3.3 Algorithm

Step 1: Local variance computation \(V(i,j)\) and determination of Vmax.

Step 2: Computation of \(k(i,j)\) inversely proportional to \(V(i,j)\) using equation [8].

Step 3: Determination of conductivity coefficients \(c\):

1- for each direction (north, south, east and west) we compute \( \|V\| \) as formulation [5]

2- for each direction, we compute \( c(\|V\|) \) using:

\[ c(\|V\|) = e^{-k_{\text{max}} \|V\|} \]

Step 4: for each iteration, we determine the output using:

\[ I_{o,i,j}^{t+1} = I_{o,i,j}^{t} + 0.2 \sum_{\text{all directions}} c(\|V_{k_{i,j}}\|) \|V_{k_{i,j}}\| \]  

(17)

with \( I_{o,i,j}^{0} \) is the original image to smooth.

The term \( \|V\| \) of equation [4] should be set to less than 0.25 to ensure a stable solution (Van den Boomgaard 2004). In this experiment, we set it to 0.2.

4. Experimental results

In this section, we describe the results of our experiments using Perona-Malik anisotropic diffusion, and the new adaptive anisotropic filter. Using 1-D and 2-D signals.

In those experiments, we use 50 iterations to compare filters in the same level.

Figure 3(a) is intensity image \(I(150 \times 150)\) corrupted with additive white Gaussian noise obeying to signal-to-noise ratio (SNR) of 25 dB. The SNR used here is calculated as:

\[ \text{SNR} = 10 \log \frac{\sum_{i=1}^{M} \sum_{j=1}^{N} I_{\text{orig}}^2(i,j)}{\sum_{i=1}^{M} \sum_{j=1}^{N}(I(i,j) - I_{\text{orig}}(i,j))^2} \]  

(18)

Where \( I_{\text{orig}} \) and \( I \) are the original and the noisy images respectively, and \( M \) and \( N \) are the number of pixels horizontally and vertically, respectively.

Figure 3(b) is the 1-D signal extracted from Figure 3(a) line 25 and Figure 3(c,d and e) are the smoothed signals of Figure 3(b) using the Perona-Malik equation.
with the scale dimension parameter $k=1, 10, 40$ respectively.

Figure 3(f) shows the result of the application of the new adaptive anisotropic filter, with $k$ is chosen adaptively from 1 to 40 inversely proportional the local intensity variance using equation [8].

The visual inspection of figure 3(c,d and e) shows that the denoising is sensitive to the value of $k$: If $k$ is small the smoothing is not performed Figure 3(c), but if $k$ is large Figure 3(e), the denoising performance is good and the edges is blurred, which is clearly undesirable if the objective is a sharp segmentation. Those problems have been surmounted with an adaptive algorithm Figure 3(f), i.e., The noise has been removed, while the depth discontinuities have obviously been preserved.

As a real sample, we use an image contained an horizontal section of backbone, Figure 4(a).

Figure 4 (b-e) show the restored images using the standard P-M filter with $k=10, 30, 80$ and 100, respectively. And Figure 4(f) is the restored one using our adaptive method with $k$ is chosen adaptively from 10 to 100. The same conclusion can be drawn from Figure 3, i.e., if $k$ is small the boundaries are preserved but the smoothing is not sufficient, and if $k$ is large, the image becomes blur and several wrong edges are detected.

In the adaptive case Figure 4(f), almost all irrelevant details are removed but sharp boundaries are preserved. Those features have a big advantage in image segmentation. After contour detection (contour command in Matlab 6) Figure 5. It is evident from those figures that the conventional method cannot reach the improvement of the proposed approach.

![Figure 3](image3.png)

**Figure 3.** Adaptive smoothing of 1-D signal: (a) Intensity image corrupted with Gaussian white noise $SNR=25$; (b) 1-D signal extracted horizontally from (a); (c, d and e) adaptive smoothed signal with $k=1, 10, 40$ respectively; (f) Adaptive smoothed signal with $k$ is determined by [8].

![Figure 4](image4.png)

**Figure 4.** Adaptive smoothing of an intensity image. (a) original image. (b, c, d and e) adaptive smoothed images with $k=10, 30, 80$ and 100 respectively. (f) Adaptive smoothed image using an adaptive $k$ determined by [8].
Figure 5. Contour detection of backbone image. (a) detected contour of the original image. (b, c, d and e) detected contour of the adaptive smoothed image with k=10, 30, 80 and 100 respectively. (f) detected of the adaptive smoothed image using an adaptive k determined by [8].

Conclusion

We have presented in this paper an Adaptive Anisotropic Diffusion technique to smooth a signal while preserving discontinuities. In P-M equation, we have proposed to adapt the scale dimension k parameter of the diffusivity function according to the local intensity variance, we have shown that this technique removes noise and irrelevant details while preserving sharper boundaries.

The proposed algorithm was compared with the classical P-M technique, and the results show that the new approach outperform P-M technique. The extension of this approach to multi-spectral images can be envisaged.

References


